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# *QUATERNIONS.*

BY PROF. DE VOLSON WOOD, HOBOKEN, NEW JERSEY.

(Continued from page 71.)

## OBLIQUE VECTORS.

26. IN section 15 of the preceding article, it was stated that, comparing vector $\beta$ with vector $\alpha$ consists of the compound operation of determining the angle through which $\alpha$ must be rotated in order to bring it into a position *parallel* to $\beta$ and of comparing the *length* of $\beta$ with that of $\alpha$.

Parallelism implies a zero angle between the *directions* considered. Thus, if $\beta$ is to be compared with $-\alpha$, then $-\alpha$ must be revolved to coincide in *direction* with $+\beta$. In section 19 a notation was adopted to represent the rotation—left handed being considered positive. The ratio of their lengths is simply a number — positive and abstract — and is expressed in the well known way; hence if $a$ and $b$ represent the lengths respectively of $\alpha$ and $\beta$, $t$ the angle between the vectors (the right angle being unity), and $i$ a positive unit vector perpendicular to the plane of $\alpha\beta$, we may write

$$\frac{\beta}{\alpha} = \frac{T\beta . U\beta}{T\alpha . U\alpha} = \frac{T\beta}{T\alpha} . \frac{U\beta}{U\alpha} = T\frac{\beta}{\alpha} . U\frac{\beta}{\alpha} = \frac{b}{a} i^t. \tag{14}$$

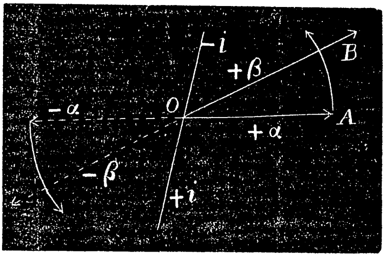
For all values of $\alpha$ and $\beta$, we have

$$U\frac{\beta}{\alpha} = i^t. \tag{15}$$

If $a = b$ equation (14) reduces to (15); hence for *unit* vectors $U$ may be omitted without ambiguity, in which case we may write

$$\frac{\beta}{\alpha} = i^t;^* \tag{16}$$

hence $\beta \div \alpha$, as unit vectors, represents a versor. (See sec. 20.)



If the operation be observed on the side of the negative axis, $-i$, the rotation of $\alpha$ into $\beta$ will be right-handed, or negative, and will be represented by $-t$; hence we have, as unit vectors

$$\frac{\beta}{\alpha} = (-i)^{-t}; \tag{17}$$

which compared with (16) gives

$$i^t = (-i)^{-t}; \tag{18}$$

that is, the *value* of a versor remains the same when the signs of its base and exponent are both changed from + to —, or the reverse.

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[*] Hamilton's Lectures, p. 127.

It is well to observe that in the expression $(-i)^t$, the negative sign applies to the vector—or axis— only; but in the expression $-i^t$ the entire versor is negative and is the same as $-(i^t)$. The MINUS in the last expression has the effect of reversing* the *direction* of the result produced by $i^t$.

The angles between the vectors will be limited to those between zero and plus or minus two right angles, unless otherwise expressly stated. The angle between $\alpha$ and $-\beta$ (see the figure) is the same as between $-\alpha$ and $\beta$, and is $2-t$ (being the supplement of the angle $t$ between vectors $+\alpha$ and $+\beta$), and is positive in reference to the negative axis $-i$; hence we have

$$\frac{-\beta}{\alpha} = \frac{\beta}{-\alpha} = -\frac{\beta}{\alpha} = -\left(\frac{\beta}{\alpha}\right) = (-i)^{2-t} = i^{t-2}. \qquad (19)$$

From (16) and (19) we find

$$i^t = -(-i)^{2-t}. \qquad (20)$$

The second member of (20) [which is equivalent to $-(-i^2 . -i^{-t})$] shows that the vector operated upon (as $\alpha$) may be rotated positively about the negative axis $-i$ through two qnadrants, followed by a negative rotation of $t$ about the same axis, thus making it coincide with the vector $(-\beta)$ which was divided, after which the result is reversed, thus bringing it into the position of $+\beta$. The left member rotates $\alpha$ into $\beta$ positively in one operation.

The value $\beta \div \alpha$ may now be written in various torms. Thus, as unit vectors, we have from the preceding equations,

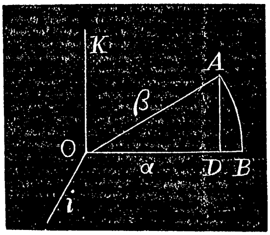
$$\frac{\beta}{\alpha} = \beta \div \alpha = i^t = (-i)^{-t} = \frac{1}{(-i)^t} = \frac{1}{i^{-t}} = -(-i)^{2-t} = -\frac{1}{i^{2-t}}, \text{ etc.} \qquad (21)$$

If vector $\beta$ be operated upon, turning it to a parallelism with $\alpha$, the rotation will be positive about $-i$ (the relative positions of the lines being as before), and we have [see (16), (18), and (20)]

$$\frac{\alpha}{\beta} = (-i)^t = i^{-t} = \frac{1}{i^t} = -i^{2-t}, \text{ etc.}; \qquad (22)$$

from which we have

$$-i^{-t} = -\frac{1}{i^t} = i^{2-t}.\dagger \qquad (23)$$



27. We now proceed to decompose the versor.

Let $\alpha$ and $\beta$ be two coinitial *unit* vectors making an angle $\theta$ between them. Take $i$, a unit vector perpendicular to the plane of $\alpha\beta$, positive in front of the plane, and $k$ perpendicular to $\alpha$ and $i$. From the extremity of $\beta$ drop the perpendicular $AD$ upon $\alpha$; then as lines we have

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*It is an INVERSOR, or its effect is an *in*-version. Lectures, p. 54. †Ibid. pp. 331, 536.

$$OD = \cos\theta, \qquad DA = \sin\theta,$$

and as vectors (§6), $DA$ being parallel to $Ok$,

$$OD = \cos\theta . a, \qquad DA = \sin\theta . k.$$

According to equation (1) we have

$$\beta = OD + DA = \cos\theta . a + \sin\theta . k. \qquad (24)$$

Dividing $a$ *into* both members of (24), observing that $k \div a$ is a quadrantal versor and positive about $i$ [Eq. (3)], we have [see also Eq. (21)]

$$\begin{aligned} \frac{\beta}{a} &= \cos\theta + \sin\theta \frac{k}{a} \\ &= \cos\theta + i \sin\theta = i^t, \qquad (25) \end{aligned}$$

where $t = \theta \div \frac{1}{2}\pi$. In a similar manner we may find directly from the figure, or by taking the reciprocal of both members of (25), observing that $i^2 = -1$,

$$\frac{a}{\beta} = \cos\theta - i \sin\theta = (-i)^t, \qquad (26)$$

the last member of which is from eq. (22). The term $i \sin\theta$ will be negative if $i$ be positive and $\theta$ negative, in which case the third member will be $i^{-t}$, as already shown in equation (22).

Multiplying $a$ *into* both members of (24) gives

$$\begin{aligned} a\beta &= a^2 \cos\theta + ak \sin\theta \\ &= -\cos\theta + i \sin\theta = i^{2-t}, \qquad (27) \end{aligned}$$

in which the exponent is $2 - t$ because $\cos\theta$ being negative and $\sin\theta$ positive, the angle of the product is the supplement if $\theta$; that is, the supplement of the angle between the vectors.

Multiplying both members of (24) into $a$ gives

$$\begin{aligned} \beta a &= a^2 \cos\theta + ka \sin\theta \\ &= -\cos\theta - i \sin\theta = (-i)^{2-t}. \qquad (28) \end{aligned}$$

The second members of the last four equations are versors. In division, the versor operating on the divisor line is conceived to turn it, positively, about the axis of the versor ($+i$, or $-i$) through an angle equal to that of the versor ($\theta$) to coincide in direction with the dividend line. In multipl'n the versor operating upon the multiplier line [as $a$, eq. (27)] is conceived to turn it in a positive direction about the axis of the versor ($+i$) through an angle equal to the *supplement* of the angle of the versor [$\pi - (\pi - \theta) = \theta$], making it coincide in direction with the multiplicand line (as $\beta$).*

The preceding equations show that the versor may be decomposed into two parts, one of which is a number equal to $\pm \cos\theta$; the other, a multiple of a *unit vector* and equal $\sin\theta . i$, the unit vector being the axis of the quotient, or of the product, of two unit vectors. Hence if $q_1$ represent the quotient (or product) of two unit vectors in space, $w_1$ the term in the result containing the *number* only, and $\rho_1$ the term containing the *vector*, we may write

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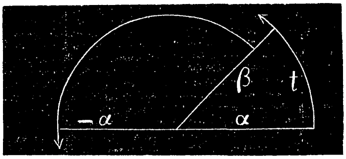
*Lectures, p. 85.

$$q_1 = w_1 + \rho_1. \tag{29}$$

28. As unit vectors we may write

$$\frac{\beta}{\alpha} = \beta\alpha^{-1} \text{ (but not } \frac{\beta}{\alpha} = \alpha^{-1}\beta). \tag{30}$$



Observing that $\alpha^{-1} = -\alpha$ [eq. (9)], the product of $\beta$ into $\alpha^{-1}$ is the same as $\beta$ into $-\alpha$, the rotation being positive, and the angle of the *product* the *supplement* of the angle between $\beta$ and $-\alpha$, that is $t$, the right angle being unity; and the *division* of $\alpha$ into $\beta$ implies a positive rotation through the same angle $t$; hence the two *results* are equal. But the product $\alpha^{-1}\beta$ *reverses* the direction of the rotation, and hence cannot equal the former result. Generally, we have

$$U\frac{\beta}{\alpha} = U\beta\alpha^{-1} = -U\beta\alpha^*.$$

Numerous other results present themselves, but we pass them for the present and proceed to the establishment of the general equations and the completion of the nomenclature.

29. If $\alpha$ and $\beta$ be not unit vectors, let their lengths be $a$ and $b$ times that of the unit, then $T\alpha = a$, $T\beta = b$, and we have

$$\left.\begin{aligned}
\frac{\beta}{\alpha} &= \frac{b}{a}\left(\cos\theta + i\sin\theta\right) = \frac{b}{a}i^t \\
\frac{\alpha}{\beta} &= \frac{a}{b}\left(\cos\theta - i\sin\theta\right) = \frac{a}{b}(-i)^t = \frac{a}{b}i^{-t} \\
\alpha\beta &= ab(-\cos\theta + i\sin\theta) = ab.i^{2-t} \\
\beta\alpha &= ab(-\cos\theta - i\sin\theta) = ab(-i)^{2-t}.
\end{aligned}\right\} \tag{A}$$

These are general equations for the division, or for the multiplication, as the case may be, of any two directed lines in space; and such quotient, or product constitutes a quaternion. By a discussion of equations ($A$) all the practical equations of this science may be deduced, including the results of the preceding article and those thus far given in the present one.

Thus if $\theta = 90°$, all the results for the multiplication, and for the division, of rectangular vectors are quickly found. The 3d and 4th give $\alpha\beta = -\beta\alpha$.

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*Lectures, p. 411.

If $\theta = 0°$ the 3rd and 4th will gives $\alpha\beta = + \beta\alpha$. If $a = b$, the 1st and 4th of ($A$) will differ only by having contrary signs, hence involving the third member in the expression, we have

$$\frac{\beta}{\alpha} = -\beta\alpha = i^t = -(-i)^{2-t}.$$

The angle between $\beta$ and $-\alpha$ is the supplement of the angle between $\alpha$ and $\beta$, the axis of rotation being positive, or $+i$; therefore for unit vectors we have

$$\beta(-\alpha) = \cos\theta + i\sin\theta,$$

which will be the same as the 1st of ($A$) when $a = b$; hence we will have

$$\frac{\beta}{\alpha} = \beta(-\alpha) = \beta\alpha^{-1},$$

as shown in eq. (30).

The parenthetical part of the second members of equations ($A$) is the versor, and we may write

$$U\frac{\beta}{\alpha} = \cos\theta + i\sin\theta = i^t, \tag{31}$$

$$U\alpha\beta = -\cos\theta + i\sin\theta = i^{2-t}; \tag{32}$$

and the 1st and 3rd of ($A$) give

$$U\frac{\beta}{\alpha} = - U\beta\alpha, \text{ as before shown.} \tag{33}$$

For a representative expression, we choose one, the elements of which are naturally positive. For this reason the 1st of ($A$) may be considered as TYPICAL — since $a$ and $b$ are essentially positive, the axis of rotation ($i$), positive, the direction of rotation, positive, and when $\theta$ is acute, $\cos\theta$ and $\sin\theta$ will be positive. If $q$ represent the left member, we have as a TYPICAL FORM

$$Uq = \cos\theta + i\sin\theta \tag{34}$$

$$= \cos\theta + \sqrt{-1}.\sin\theta,$$

the last value of which is independent of any particular axis, and is suggestive of De Moivre's formula.

30. The second members of equations ($A$) are composed of two parts, one of which is a *number* equal to a multiple of $\cos\theta$, the other is a multiple of a unit vector, as $ab \sin \theta.i$; hence if $q$ represent the first member, $w$ the *numerical term* and $\rho$ the entire vector in the second term of the second member, we have the typical form

$$q = w + \rho,^* \tag{35}$$

which reduces to (29) when $\alpha$ and $\beta$ are each unit vectors.

31. A SCALAR is a number, either positive or negative—it represents the *reals* of algebra. This name is applied to such quantities because they

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*Lectures, p. 116.

may be laid off on a scale from $-\infty$ to $+\infty$.[*] It is indicated by the letter $S$. The numerical term in equations ($A$) is called the scalar part[†] of the quotient, or product (as the case may be) of two directed lines. Hence we have

$$S.\frac{\beta}{\alpha} = \frac{b}{a}\cos\theta, \qquad S.\frac{\alpha}{\beta} = \frac{a}{b}\cos\theta, \tag{36}$$

$$S.\beta\alpha = -ab\cos\theta, \qquad S.\alpha\beta = -ab\cos\theta. \tag{37}$$

The last is read "The scalar of the product of $\alpha$ into $\beta$ equals minus the continued product of tensor $\alpha$, tensor $\beta$, and the cosine of the angle between $\alpha$ and $\beta$"—and similarly for the others. The period after $S$ indicates that the scalar of all that follows it in that term is to be taken. It may be omitted where no ambiguity will result, hence the preceding expressions may be written $S\frac{\beta}{\alpha}$, $S\alpha\beta$, etc.

From (37) we have

$$S\alpha\beta = S\beta\alpha. \tag{38}$$

From 3rd and 4th of ($A$), we have

$$\alpha\beta + \beta\alpha = 2S\alpha\beta. \tag{39}$$

From (31), (32), (33), and ($A$), we have

$$SU\frac{\beta}{\alpha} = \cos\theta\text{[‡]} = SU\frac{\alpha}{\beta} = -SU\beta\alpha,\text{[**]} \tag{40}$$

$$SU\alpha\beta = -\cos\theta = SU\beta\alpha.\text{[††]} \tag{41}$$

The scalar of a number is the same number, hence

$$SSq = S^2q = Sq = \pm\cos\theta,$$

or briefly $S^2 = S.$[‡‡] And, generally,

$$Sq = Tq.SUq. \tag{42}$$

32. The second term of the second member of equations ($A$) is called the VECTOR PART[‡‡] of the quaternion and is indicated by the letter $V$. Thus

$$V\frac{\beta}{\alpha} = \frac{b}{a}\sin\theta.i, \qquad V\frac{\alpha}{\beta} = -\frac{a}{b}\sin\theta.i; \tag{43}$$

$$V\alpha\beta = ab\sin\theta.i, \qquad V\beta\alpha = -ab\sin\theta.i; \tag{44}$$

$$\therefore\ V\alpha\beta = -V\beta\alpha. \tag{45}$$

From 3rd and 4th of ($A$) we have

$$\alpha\beta - \beta\alpha = 2V\alpha\beta. \tag{46}$$

From (32) we have

$$VU\alpha\beta = \sin\theta.i. \tag{47}$$

Observing that $i$, unaffected by an exponent, is a quadrantal versor (§20), the versor of $\sin\theta\,.\,i$ is $i$; hence from (47) we have

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[*] Lectures, p. 58. [†] Ibid., p. 392. [‡] Ibid., p. 397. [**] Ibid., p. 412. [††] & [‡‡] Ibid., p. 393.

$$UVU\alpha\beta = i. \tag{48}$$

Similarly from (44) we have

$$UV\alpha\beta = i = -\,UV\beta\alpha, \tag{49}$$

which value is the same as (48). In (44), $ab\sin\theta$ is the tensor of $i$ (or of $-\,i$), hence

$$TV\alpha\beta = TV\beta\alpha = ab\sin\theta.^{*} \tag{50}$$

Similarly (32) gives

$$TVU\alpha\beta = \sin\theta.^{\dagger} \tag{51}$$

Equation (50) shows that $TV\alpha\beta$ is the area of a parallelogram constructed on $T\alpha$ and $T\beta$ as adjacent sides having an angle $\theta$ between them.

From (43) we have $V\frac{\beta}{\alpha}\,.\,V\frac{\alpha}{\beta} = \sin^2\theta$. Numerous other relations may be readily deduced from the preceding equations.

33. According to the preceding notation, equations ($A$) may be written in the following form:

Equations ($A$) [see (14)] give $q = Tq.Uq$; (52)
" (36), (37), (43), (44), $= Sq + Uq$; (53)
" (36), (37), (40), (43), (44) $= Tq.SUq + Tq.\sin\theta.i$;
factoring $= Tq(SUq + \sin\theta.i)$,
substituting (47) $= Tq(SUq + VUq)$, (54)
or substituting (51) $= Tq(SUq + TVUq.i)$,
in which substitute (49) $= Tq(SUq + TVUq.UVq)$,
or subst. (48) instead of (49) $= Tq(SUq + TVUq.UVUq)$.

NOTE.—For § 9, page 36, substitute the following:

9. A TENSOR is a numerical factor by which a unit vector is multiplied to produce a given vector. Literally the term implies a stretching, but its meaning is here enlarged so as to include the shortening of a line. It is the *ratio* of the *length* of a vector to that of a unit parallel vector, and hence is always positive, or more strictly speaking, *signless*, and may be an entire, fractional, or incommensurable number. The tensor of a vector is sometimes indicated by the letter $T$, as $T\gamma$. Thus in the equations $\gamma = b\gamma_1$, $\delta = -\,c\delta_1$, $\eta = 3\eta_1$, where $\gamma_1$, $\delta_1$, $\eta_1$, are unit-vectors, we have $T\gamma = b$, $T\delta = T(-\delta) = c$, $T\eta = 3$. Generally, we have

$$\gamma = T\gamma(\gamma_1).$$

[For want of sorts, $\gamma$, $\delta$, and $\eta$ are used in the foregoing Note, respectively instead of *Alpha*, *Beta* and *Gamma*, used by Prof. Wood.]

NOTE 2.—For the following corrections I am indebted to suggestions by Alex. S. Christie, communicated to me by the editor.

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*Lectures, p. 418. †Ibid., p. 397.

At page 67, after the last line, add, When the direction of rotation of the fraction and its reciprocal are in opposite senses, the reciprocal of the axis is the axis of the reciprocal, as may be shown by equation (9).

At page 69, lines 7 and 8, expunge the sentence "As a versor it implies that $\beta$ has been turned from some arbitrary direction into the given one."; for it conflicts with the definition given by Hamilton, which is "$U\beta$ is *the versor of a right quotient.*" (Elements, p. 136.)

At page 70, line 3, for "has a fixed position," read, is confined to one plane.

Also, at page 68, line 3, for "$i \div -i$" read, $-i \div i$; and at page 71, line 16, for "$-kij = -ijk = -jki$" read, $-kij = -ikj = -jik$.

[Prof. Wood proposes to discuss some examples illustrating the foregoing principles, which will appear in future Nos., as space will permit.]

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## *A FOURTH PROPORTIONAL.*

BY JESSE APPLEGATE, YONCALLA, OREGON.

If a straight line, $ABC$, be considered as composed of two parts, $a$ and $b$, or the line $a$ be laid upon the line $b$, the lines $a$ and $b$, in neither case, form an angle at $B$, for the lines $a$ and $b$ are *extended* in a line with each other, and their relation to each other at their junction $B$, would be expressed as an angle of 180°, or a right line. In the second case, as the line $a$ coincides with the line $b$, the angle at $B$ may be represented by 0°.

Whatever the lengths of the lines $a$ and $b$ may be, a *proportion* exists of which $a$ and $b$ are the means, and their sum $a + b$, one of the extremes. That is,

$$a + b : b = a : x,$$

whence

$$x = \frac{ab}{a + b},$$

$x$ being the 4th proportional.

In the above cases, where $a$ and $b$ form a straight line, the 4th proportional depends on the *lengths* of the lines only, it will therefore also exist whatever *angle* these lines make with each other, from 180° to 0°.

To ascertain the utility, if there be any, in the 4th proportional; let the point $B$ be the apex, and carry the line $a$ around until it coincides with the line $b$, stopping at any point, say at 90°, or before or after passing 90°, to inquire what value the 4th proportional may have in solving problems in Geometry and Trigonometry.